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A Logical Description for Perfect Measurements[†]

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We reconsider the description for property transitions due to perfect measurements, viewing them as a special case of general transitions that are due to an externally imposed change. We propose a corresponding syntax involving operational quantum logic and a fragment of noncommutative linear logic.

1. INTRODUCTION

In the spirit of Piron [7, 9, 10], Jauch and Piron [6], Aerts [12], Foulis *et al.* [13], Foulis and Randall [14], Faure *et al.* [21], Moore [22, 27], Amira *et al.* [24], and Coecke and Stubbe [25, 26], we will point out the correspondence between the act of 'inducing properties' [24] and perfect measurements [9], that is, a joint ideal measurement of the first kind of a property and its orthocomplement. We describe the transitions that occur in such a perfect measurement, and this will involve aspects of algebraic quantum logic [1, 3, 5, 7, 15] and linear logic [17, 18] in particular some of the noncommutative variants [19, 20]. Indeed, since general not-necessarily deterministic property transitions yield quantale descriptions [24, 26], whereas noncommutative linear logic yields quantale semantical models, the formal motivation for a logical description of perfect measurements incorporating linear logical operations naturally arises.

In this paper, operational quantum logic (OQL) stands for the Geneva school approach to states and properties of a physical entity; we will not go into details and refer for the most recent overview to Moore [27]. The

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properties $\mathcal{L}$ of the entity are ordered by a—physically deducible—implication relation, as such structured as a poset which *proves* to be meet-complete as a semilattice [19], and as such also complete as a lattice, the join given by $\vee_i a_i = \wedge\{a \in \mathcal{L} | \forall i\colon a \geq a_i\}$. A property is said to be actual if it is *true*, i.e., any verification of it would yield a positive answer with certainty, and a state is then defined as the strongest actual property an entity possesses. We call a set of properties an *actuality set* if at least one property in it is actual. Note that if A is an *actuality set*, we have immediately that $\vee A := \vee_{a \in A} a$ is an actual property, and we will refer to this 'strongest property of which A implies actuality' as a *definite actual property* of an actuality set. As shown by Piron [7, 9], a complete lattice yields a representation as the projection lattice of a 'generalized' Hilbert space if and only if it is atomistic, orthomodular, and satisfies the covering law, as such assuring a realization within standard quantum theory. In this paper it suffices to require, besides completeness, only orthomodularity of the lattice, the latter defined by (i) there exists ${}^\perp\colon \mathcal{L} \to \mathcal{L}$ fulfilling $a \leq b \Rightarrow b^\perp \leq a^\perp$, $a \wedge a^\perp = 0$, $a \vee a^\perp = 1$, $a^{\perp\perp} = a$, and (ii) $a \leq b$ implies $a \vee (a^\perp \wedge b) = b$. In Section 2 we show how the scheme developed in refs. 24–26 'lifts' the Baer*-semigroups considered as state transitions by Pool [8] —and introduced by Foulis [4] as a natural collection of morphisms for orthomodular lattices that embeds the closed orthogonal projections on this lattice—to more general classes of morphisms that express indeterministic transitions. In Section 3 we translate this into logical axioms, explicitly expressing that these transitions are due to the interaction with an externally imposed context, and, in the particular case of quantum measurements, a 'perfect measurement context'.

2. PROPAGATION OF PROPERTIES DUE TO A PERFECT MEASUREMENT

The maps $\mathcal{P}(\mathcal{L}) = \{\varphi_a\colon \mathcal{L} \to \mathcal{L}\colon b \mapsto a \wedge (b \vee a^\perp) | a \in \mathcal{L}\}$—the 'Sasaki [3] projections'—prove to be the set of all closed orthogonal projections on $\mathcal{L}$, and are by $\theta\colon \mathcal{L} \to \mathcal{P}(\mathcal{L})\colon a \mapsto \varphi_a$ in isomorphic correspondence with $\mathcal{L}$ when ordered by $\varphi_a \trianglelefteq \varphi_{a'} \Leftrightarrow \varphi_a\varphi_{a'} = \varphi_a$ [4, 23] However, the maps φ_a are not closed under their natural operation 'composition', and should be considered as embedded in $\mathcal{S}(\mathcal{L}) := \{f\colon \mathcal{L} \to \mathcal{L} | f(\vee_i a_i) = \vee_i f(a_i)\}$, the corresponding complete Baer*-semigroup: note that here $\mathcal{S}(\mathcal{L})$ is itself a join-complete lattice with respect to the pointwise computed order $f \leq f' \Leftrightarrow \forall a \in \mathcal{L}$: $f(a) \leq f'(a)$, yielding a pointwise computable join for all $\{f_i\}_i \subseteq \mathcal{S}(\mathcal{L})$ as $\vee_i f_i\colon \mathcal{L} \to \mathcal{L}\colon a \mapsto \vee_i f_i(a)$. However, the inclusion of $\mathcal{P}(\mathcal{L})$, with ordering inherited from $\mathcal{L}$ through θ, in $(\mathcal{S}(\mathcal{L}), \vee)$ does not preserve the partial order. Indeed, for $a, a' \in \mathcal{L}$ with $a' \not\leq a^\perp \not\leq 1$ and $a \wedge a' = 0$ we have $a \leq a \vee a'$, but since $0 < \varphi_a(a') \leq a$ and $\varphi_{a\vee a}{}'(a') = a'$ it follows that $\varphi_a' \wedge$

$\varphi_{a\vee a}'(a') = 0$ and thus $\varphi_a(a') \not\leq \varphi_{a\vee a}'(a')$, so that $\varphi_a \not\leq \varphi_{a\vee a}'$, although $\varphi_a \trianglelefteq \varphi_{a\vee a}'$. Instead of considering the closed orthogonal projections $\mathcal{P}(\mathcal{L})$, we will consider $\mathcal{P}^{\#}(\mathcal{L}) = \{\varphi_{\{a,a^\perp\}} | a \in \mathcal{L}\}$ with ($\mathcal{P}\mathcal{L}$ denotes the powerset of $\mathcal{L}\setminus\{0\}$)

$$\begin{aligned}\varphi_{\{a,a^\perp\}}: \quad \mathcal{P}\mathcal{L} \to \mathcal{P}\mathcal{L}: \quad B \mapsto & \{\varphi_a(b) | b \in B, b \not\leq a^\perp\} \\ & \cup \{\varphi_{a^\perp}(b) | b \in B, b \not\leq a\}\end{aligned} \tag{1}$$

The maps in $\mathcal{P}^{\#}(\mathcal{L})$ will be interpreted as describing the propagation of properties in perfect measurements:

- A property $b \in \mathcal{L}$ that is actual before the measurement yields an actual property $\varphi_a(b)$ or $\varphi_{a^\perp}(b)$ after it. In general neither $\varphi_a(b)$ nor $\varphi_{a^\perp}(b)$ 'will be' actual with certainty after the measurement, provided that $b \not\leq a$ and $b \not\leq a^\perp$. The strongest 'definite actual property' for the actuality set $\{\varphi_a(b), \varphi_a^\perp(b)\}$ is $\varphi_a(b) \vee \varphi_a^\perp(b)$. It has been motivated that actual properties propagate preserving the join, as such giving the maps in $\mathcal{S}(\mathcal{L})$ the significance of describing the propagation of actual properties. Therefore, given an initial actuality set, we consider them as describing the propagations of the definite actual properties.

It is again natural to consider the maps $\varphi_{\{a,a^\perp\}}$ as belonging to a more general collection closed under composition, but due to the formal change of domain from the $\vee$-lattice $\mathcal{L}$ to the $\cup$-lattice $\mathcal{P}\mathcal{L}$, this requires an essentially different approach. A construction that allows this embedment is proposed in refs. 24–26 and will now be applied to the maps defined by (1). The set $\mathcal{P}^{\#}(\mathcal{L})$ is canonically in surjective correspondence with $\mathcal{P}(\mathcal{L})$ and thus with $\mathcal{L}$ itself by ϕ: $\mathcal{P}(\mathcal{L}) \to \mathcal{P}^{\#}(\mathcal{L})$: $\varphi_a \mapsto \varphi_{\{a,a^\perp\}}$ that factorizes in

$$\begin{cases}\theta^P: \mathcal{P}(\mathcal{L}) \to \mathcal{P}^P(\mathcal{L}): \varphi_a \mapsto [\mathcal{P}\mathcal{L} \to \mathcal{P}\mathcal{L}: B \mapsto \{\varphi_a(b) | b \in B, b \not\leq a^\perp\}] \\ \eta: \mathcal{P}^P(\mathcal{L}) \to \mathcal{P}^{\#}(\mathcal{L}): \theta^P(\varphi_a) \mapsto \theta^P(\varphi_a) \cup \theta^P(\varphi_a^\perp)\end{cases} \tag{2}$$

where $\mathcal{P}^{\mathrm{P}}(\mathcal{L})$ is implicitly defined as the range of θ^P. The following is obvious.

Proposition 1. (i) $\varphi_{\{a,a^\perp\}} = \varphi_{\{a^\perp,a^{\perp\perp}\}}$; (ii) $\phi(\varphi_a) = \phi(\varphi_b) \Leftrightarrow b \in \{a, a^\perp\}$.

Thus, $\mathcal{P}^{\#}(\mathcal{L})$, $\mathcal{P}(\mathcal{L})/\sim$, and $\mathcal{L}/\sim$ are in bijective correspondence for the equivalence relation $\varphi_b \sim \varphi_a$ (respectively $a \sim b$) iff $b \in \{a, a^\perp\}$. Clearly, the ordering of $(\mathcal{P}(\mathcal{L}), \trianglelefteq)$ is in no way inherited by $\mathcal{P}^{\#}(\mathcal{L})$. However, as we will show next, the maps in $\mathcal{P}^{\#}(\mathcal{L})$ can be considered as incomparable with respect to the partial ordering inherited from $(\mathcal{S}(\mathcal{L}), \leq)$. For maps f: $P\mathcal{L} \to$

$P\mathcal{L}$, denote the condition $\forall A, B \in P\mathcal{L}: \vee A = \vee B \Rightarrow \vee f(A) = \vee f(B)$ as $A^{\#}$, and define the following subset of $\mathcal{S}(P\mathcal{L})$:

$$\mathcal{Q}^{\#}(\mathcal{L}) := \{ f: P\mathcal{L} \to | f(\cup_i a_i) = \cup_i f(a_i), f \text{ meets } A^{\#}\} \tag{3}$$

As shown in refs. 24–26, it is exactly the condition $A^{\#}$ that forces the definite actual properties to propagate preserving the join. We will now sketch how this connection between $Q^{\#}(\mathcal{L})$ and $\mathcal{S}(\mathcal{L})$ is realized. Set

$$\theta^P: \quad \mathcal{S}(\mathcal{L}) \to \mathcal{S}(\mathcal{P}\mathcal{L}): \quad f \mapsto [\mathcal{P}\mathcal{L} \to \mathcal{P}\mathcal{L}: \mathcal{B} \mapsto \{ f(b) | \, b \in \mathcal{B}\}] \tag{4}$$

Recall that a quantale [16] is a join-complete lattice equipped with an operation $\circ$ that distributes at both sides over arbitrary joins. Quantale morphisms preserve $\circ$ and all joins. The following proposition can be found in Coecke and Stubbe [25, 26].

Proposition 2. The set $P\mathcal{S}^P(\mathcal{L}) = \{\cup_i \theta^P(f_i) | \, \forall i: f_i \in \mathcal{S}(\mathcal{L})\}$ defines a strict subquantale of $(\mathcal{Q}^{\#}(\mathcal{L}), \cup, \circ)$, where $\cup$ denotes pointwisely computed unions and $\circ$ composition. Moreover, the map $\bigvee[-]: \mathcal{Q}^{\#}(\mathcal{L}) \to \mathcal{S}(\mathcal{L}): f \mapsto [\mathcal{L} \to \mathcal{L}: a \mapsto \vee f(\{a\})]$ is a surjective quantale morphism.

Clearly $\mathcal{P}^{\#}(\mathcal{L}) \hookrightarrow P\mathcal{S}^P(\mathcal{L})$, which is closed under composition, and since $\mathcal{P}^{\#}(\mathcal{L}) \hookrightarrow \mathcal{Q}^{\#}(\mathcal{L})$, the corresponding definite actual properties propagate preserving the join. Indeed, given the union-preserving map $\varphi_{\{a,a^{\perp}\}}$, the map describing the propagation of definite actual properties is the join-preserving map $\vee[\varphi_{\{a,a^{\perp}\}}]:\mathcal{L} \to \mathcal{L}: b \mapsto \varphi_a(b) \vee \varphi_{a'}(b)$. The following scheme summarizes all the above situating $\mathcal{P}(\mathcal{L})$ and $\mathcal{S}(\mathcal{L})$ relative to $\mathcal{P}^{\#}(\mathcal{L})$ and $\mathcal{Q}^{\#}(\mathcal{L})$:

$$\begin{array}{lllllll} \mathcal{L} \overset{\cong}{\leftrightarrow} \mathcal{P}(\mathcal{L}) & \hookrightarrow & (\mathcal{S}(\mathcal{L}), \vee, \circ) & & & & \\ \quad\ \downarrow \phi & & & \nwarrow^{\vee[-]} & & & \\ \quad\ \mathcal{P}^{\#}(\mathcal{L}) & \hookrightarrow & (P\mathcal{S}^P(\mathcal{L}), \cup, \circ) & \hookrightarrow & (\mathcal{Q}^{\#}(\mathcal{L}), \cup, \circ) & \hookrightarrow & (\mathcal{S}(\mathcal{P}\mathcal{L}), \cup, \circ) \end{array} \tag{5}$$

We will now apply the above to the physical situation where a physical entity is placed in a measurement context that induces a perfect measurement. Recall that $a, b \in \mathcal{L}$ are compatible if the Boolean sublattice generated by $\{a, a^{\perp}, b, b^{\perp}\}$ distributes.

Definition 1. An *induction* on a physical entity is an externally imposed change of properties. A *perfect measurement induction*, characterized by a pair $\{a, a^{\perp}\}$, is an induction such that after it the property a is actual 'or' the property $a^{\perp}$ is actual, and any property b which is compatible with a—and as such also with $a^{\perp}$—that is actual before is still actual afterward.

Using (1), Proposition 1, and Theorem 4.3 of ref. 9, p. 69, one obtains:

Proposition 3. Given a perfect measurement induction of $\{a, a^{\perp}\}$ on an entity with property lattice $\mathcal{L}$, the map $\varphi_{\{a,a^{\perp}\}}: P\mathcal{L} \to P\mathcal{L}$ describes the propagation of actuality sets.

Clearly, these perfect measurement inductions can be interpreted as a minimal disturbance of the entity assuring actuality of a 'or' $a^{\perp}$. The essence of this remark boils down to the use of 'or', which in this case is a disjunction expressing what we could call inner nondeterminism:

- Indeed, crucial in the notion of a perfect measurement induction is that we express how two properties both have an ability 'to be' actual in case we are going to perform the induction. This aspect forces us to consider unions of transitions and underlying actuality sets, i.e., considering maps in $\mathfrak{Q}^{\#}(\mathcal{L})$ rather than in $\mathcal{S}(\mathcal{L})$.

3. A LOGICAL DESCRIPTION FOR PERFECT MEASUREMENTS

In this section we give logical axioms for the propagation of properties, more precisely, of actuality sets, in perfect measurements—i.e., the maps above described by $\mathcal{P}^{\#}(\mathcal{L})$ and their finite compositions—explicitly expressing that the measurement process is a nondeterministic transition due to interaction with an externally imposed context [e.g., 11]. A detailed description of the logical language, containing the elementary and well-formed formulas linked to physical entities, and sequent calculus for describing property transitions in general is elaborated on in ref. 28. In this paper we stick to a combination of a fragment of noncommutative linear logic and OQL. We will need some of the Left and Right sequent rules of noncommutative intuitionistic linear logic *NCIL* as defined in ref. 19, extended to a predicate calculus:

$$(id.) := \overline{A \vdash A} \qquad (cut) := \frac{\Gamma \vdash A \quad \Gamma_1, A, \Gamma_2 \vdash \Delta}{\Gamma_1, \Gamma, \Gamma_2 \vdash \Delta}$$

$$(\otimes, R) := \frac{\Gamma_1 \vdash A \quad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \otimes B} \qquad (\otimes, L) := \frac{\Gamma_1, A, B, \Gamma_2 \vdash \Delta}{\Gamma_1, (A \otimes B), \Gamma_2 \vdash \Delta}$$

$$(\otimes, R1) = \frac{\Gamma \vdash A}{\Gamma \vdash (A \otimes B)} \qquad (\otimes, R2) := \frac{\Gamma \vdash B}{\Gamma \vdash (A \otimes B)}$$

$$(\otimes, L) := \frac{\Gamma_1, A, \Gamma_2 \vdash \Delta \quad \Gamma_1, B, \Gamma_2 \vdash \Delta}{\Gamma_1, (A \otimes B) \Gamma_2 \vdash \Delta}$$

$$(\multimap, R) := \frac{A,\Gamma \vdash B}{\Gamma \vdash A \multimap B} \qquad (\multimap, L) := \frac{\Gamma \vdash A \ \ \Gamma_1, B, \Gamma_2 \vdash \Delta}{\Gamma_1, \Gamma, (A \multimap B), \Gamma_2 \vdash \Delta}$$

$$(\forall, R) := \frac{\Gamma \vdash A}{\Gamma \vdash \forall x A} \ (x \textit{ not free in } \Gamma) \qquad (\forall, L) := \frac{\Gamma_1, A[y/x], \Gamma_2 \vdash B}{\Gamma_1, \forall x A, \Gamma_2 \vdash B}$$

We express the properties in $\mathscr{L}_r \backslash \{0\}$ for physical entity r as property terms (constants and variables) in our logical language. We will limit ourselves to the following primitive propositions:

$$In_r(x) := x \textit{ is actual for physical entity } r$$

$$R_r(x) := x \textit{ and only } x \textit{ is reachable for physical entity } r$$

$$M_r(x, x^\perp) := \ x \textit{ or } x^\perp \textit{ will be induced on physical entity } r$$

where formula $M_r(x, x^\perp)$ should be thought of as representing the measurement context imposed on the entity in order to induce properties, and thus stands for the 'induction' itself.

Given the fact that we work with OQL yielding an additional structure for our property terms within the considered fragment of noncommutative linear logic, we add the axioms which show essential OQL features:

$$\otimes_{x \in X} [In_r(x)] \vdash In_r(\wedge X) \qquad In_r(x) \vdash In_r(x \vee y)$$

All the following axioms express the content of an arbitrary map $\varphi^\perp_{\{a,a\}} \in \mathscr{P}^\#(\mathscr{L})$ by considering (1) and incorporate explicitly the role of the induction context:

$$\begin{cases} \underline{Trans}: \forall y \forall z: In_r(y) \otimes R_r(z) \multimap In_r(\varphi_z(y)) \otimes R_r\ (\varphi_z(y)) \\ \underline{Adjust1}: \forall x \forall y\ \{\not\leq x, \not\leq x^\perp\{: M_r(x, x^\perp) \otimes [In_r\ (y) \otimes R_r(y)] \multimap In_r(y) \\ \qquad \otimes ((R_r(x) \oplus R_r(x^\perp)) \\ \underline{Adjust2}: \forall x \forall y \leq x: M_r(x, x^\perp) \otimes [In_r(y) \otimes R_r(y)] \multimap In_r(y) \otimes R_r(y) \end{cases}$$

The first axiom *Trans* expresses the induction of one reachable property according to the Sasaki projection. The second axiom *Ajust1* expresses a readjustment of the entity relative to the imposed induction context. In other words, it expresses the act of imposing the induction context $M_r(x, x^\perp)$ on the entity r consisting of an actual and reachable property.

Consider in the following application an initial situation $M(b, b^\perp) \otimes [In(a) \otimes R(a)]$ of a physical entity with actual property a, assuming that $b \not\leq a$ and $b \not\leq a^\perp$, we drop the subscript r since we consider only one physical entity. Note that we use not only the expression of the current property which is actual for the entity, but also the fact that this same property is reachable for the entity. It is now our aim to express the fact that the entity with property a will end up with actually having one of the possible properties b or $b^{\perp}$ which are reachable according to the given induction:

$$M(b, b^{\perp}) \otimes [In(a) \otimes R(a)] \vdash [In(\varphi_b(a)) \otimes R(\varphi_b(a))] \\ \otimes [In(\varphi_{b^{\perp}}(a)) \otimes R(\varphi_{b^{\perp}}(a))] \tag{6}$$

This is indeed an expression in our fragment of noncommutative linear logic and OQL which is provable according to the following subproofs, which we can establish using our axioms and logical tools:

1. We need to adjust our entity to the given measurement context (update R, given M), using <u>*Adjust*</u>l. We need the elimination of the universal quantifier twice, and modus ponens '$A, A \multimap B \vdash B$':

$$M(b, b^{\perp}) \otimes [In(a) \otimes R(a)] \vdash In(a) \otimes (R(b) \oplus R(b^{\perp})) \tag{7}$$

2. Using distributivity of $\otimes$ over $\oplus$,[4] we obtain

$$In(a) \otimes (R(b) \oplus R(b^{\perp})) \vdash [In(a) \otimes R(b)] \oplus [In(a) \otimes R(b^{\perp})] \tag{8}$$

3. Using the axiom <u>*Trans*</u>, the elimination of the universal quantifier twice, and modus ponens, we obtain

$$In(a) \otimes R(b) \vdash In(\varphi_b(a)) \otimes R(\varphi_b(a)) \tag{9}$$

4. Using the axiom <u>*Trans*</u>, the elimination of the universal quantifier twice, and modus ponens, we find

$$In(a) \otimes R(b^{\perp}) \vdash In(\varphi_{b^{\perp}}(a)) \otimes R(\varphi_{b^{\perp}}(a)) \tag{10}$$

Finally, by the previous subproofs we obtain our goal, i.e., (6).

Some additional remarks

(i) In our formalism we can perform a succession of inductions. Referring back to (1), we can rediscover $\varphi_{\{c,c^{\perp}\}}\varphi_{\{b,b^{\perp}\}}(\{a\}) = \{\varphi_c(\varphi_b(a)), \varphi_{c^{\perp}}(\varphi_b(a)), \varphi_c(\varphi_{b^{\perp}}(a)), \varphi_{c^{\perp}}(\varphi_{b^{\perp}}(a))\}$ as, for those c that satisfy it,

$$M(c, c^{\perp}) \otimes ([In(\varphi_b(a)) \otimes R(\varphi_b(a))] \oplus [In(\varphi_{b^{\perp}}(a)) \otimes R(\varphi_{b^{\perp}}(a))]) \\ + [In(\varphi_c(\varphi_b(a))) \otimes R(\varphi_c(\varphi_b(a)))] \oplus [In(\varphi_{c^{\perp}}(\varphi_b(a))) \otimes R(\varphi_{c^{\perp}}(\varphi_b(a)))] \\ \oplus [In(\varphi_c(\varphi_{b^{\perp}}(a))) \otimes R(\varphi_c(\varphi_{b^{\perp}}(a)))] \oplus [In(\varphi_{c^{-}}(\varphi_{b^{\perp}}(a))) \otimes R(\varphi_{c^{\perp}}(\varphi_{b^{\perp}}(a)))]$$

Remark that $M(c, c^{\perp}) \otimes ([In(\varphi_b(a)) \otimes R(\varphi_b(a))] \oplus [In(\varphi_{b^{\perp}}(a)) \otimes R(\varphi_{b^{\perp}}(a))])$ could also have been expressed by $M(c, c^{\perp}) \otimes (M(b, b^{\perp}) \otimes$

[4] See Abrusci [19], in Definition 2.1, "$\langle X, \leq, 1, \perp, \top, \otimes, \&, \oplus, \multimap, \multimapinv \rangle$ is a noncommutative intuitionistic linear structure iff . . . (xvii) $\forall x \in X \forall y \in X \forall z \in X$: $z \otimes (x \oplus y) = (z \otimes x) \oplus (z \otimes y)$."

$[In(a) \otimes R(a)])$, which points to the fact that we have to restrict associativity for the multiplicative conjunction.

(ii) This logical description generalizes to all transitions of properties described in $\mathfrak{Q}^{\#}(\mathcal{L})$. Set

$\mathrm{IND}_r(\alpha)$:= *The physical entity r is placed within the context inducing propagation* α where α takes values $f\colon P\mathcal{L} \to P\mathcal{L}$ in $\mathfrak{Q}^{\#}(\mathcal{L})$, and set $K_f = \{a \in \mathcal{L} | f(\{a\}) = \emptyset\}$. The following axiom expresses the content of the maps in $\mathfrak{Q}^{\#}(\mathcal{L})$:

$$\underline{\textit{General Propagation}}: \quad \forall\alpha\forall x \notin K_\alpha: \quad IND_r(\alpha) \otimes In_r(x) \multimap \bigoplus_{z\in\alpha(\{x\})} In_r(z)$$

yielding formal implementation of $A^{\#}$ as $IND_r(\alpha) \otimes In_r(\vee X) \multimap In_r(\vee\alpha(X))$ implicity for $\alpha(X) \neq \emptyset$. We recover the combined axioms *Trans*, *Adjust*1, and *Adjust2* for a particular $\varphi_{\{y,y^\perp\}}$ as

$$\forall x: \quad IND_r(\varphi_{\{y,y^\perp\}}) \otimes In_r(x) \multimap \bigoplus_{z\in\varphi_{\{y,y^\perp\}}(\{x\})} In_r(z)$$

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